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Unsteady disturbances of streaming motions around bodies

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For small-amplitude vortical and entropic unsteady disturbances of potential flows, Goldstein proposed a partial splitting of the velocity field into a vortical part $\boldsymbol{u}^{(1)}$ that is a known function of the imposed upstream disturbance and a potential part $\nabla\phi$ satisfying a linear inhomogeneous wave equation with a dipole-type source term. The present paper deals with flows around bodies with a stagnation point. It is shown that for such flows $\boldsymbol{u}^{(1)}$ becomes singular along the entire body surface and its wake and as a result $\nabla\phi$ will also be singular along the entire body surface. The paper proposes a modified splitting of the velocity field into a vortical part $\boldsymbol{u}^{(\mathrm{R})}$ that has zero streamwise and normal components along the body surface, an entropy-dependent part and a *regular* part $\nabla\phi^*$ that satisfies a linear inhomogeneous wave equation with a modified source term.

For periodic disturbances, explicit expressions for $\boldsymbol{u}^{(\mathrm{R})}$ are given for three-dimensional flows past a single obstacle and for two-dimensional mean flows past a linear cascade. For weakly sheared flows, it is shown that if the mean flow has only a finite number of isolated stagnation points, $\boldsymbol{u}^{(\mathrm{R})}$ will be finite along the body surface. On the other hand, if the mean flow has a stagnation line along the body surface such as in two-dimensional flows then the component of $\boldsymbol{u}^{(\mathrm{R})}$ in this direction will have a logarithmic singularity.

For incompressible flows, the boundary-value problem for ϕ^* is formulated in terms of an integral equation of the Fredholm type. The theory is applied to a typical bluff body. Detailed calculations are carried out to show the velocity and pressure fields in response to incident harmonic disturbances.

1. Introduction

Streaming motions with uniform upstream conditions have been extensively studied in fluid mechanics. At high speed, the Reynolds number associated with the flow is large and the effects of viscosity are confined to only certain regions of the flow (boundary-layers, wakes, etc.) outside of which the fluid can be treated as inviscid. Moreover, for flows without shock waves the uniform upstream conditions lead to irrotational motion in the outer inviscid region. In many instances there are steady or unsteady disturbances imposed on the uniform upstream flow conditions. The most common are due to flow turbulence while others are caused by the presence of certain boundaries or by the interaction with structural elements. These disturbances invariably produce significant changes in the transport properties of the fluid near the body and often completely alter the entire flow pattern by causing instability and separation.

An important example of streaming motions with small non-uniform and non-steady disturbances is encountered in unsteady aerodynamics – a discipline concerned with the high-speed motion of a streamlined body (airfoil) in a non-uniform flow. When the flow generated by the undisturbed motion has no strong shock waves, it can be well approximated by an irrotational flow except in the thin boundary layer surrounding the body and the wake. Because of the relative high speed of the fluid motion, it is reasonable to assume in many instances that the flow disturbances are carried by the potential flow and to neglect the interaction between the disturbances themselves, and their eventual decay. This leads to the linearization of the flow field with respect to the potential flow, which is often a common feature to many mathematical treatments in unsteady aerodynamics.

Until recently, unsteady aerodynamics dealt primarily with flat plate airfoils and hence the governing equations of the flow were linearized about a uniform parallel mean flow. The standard mathematical method for obtaining solutions to these equations consisted of splitting the velocity field into solenoidal and irrotational parts. The former represents a purely convected vorticity wave whose mathematical form is readily determined from the disturbed upstream conditions. The irrotational part satisfies a constant-coefficient homogeneous wave equation which reduces to a Laplace equation for incompressible flows. The first solution of this kind was obtained by Sears (1941) for a flat plate moving in an incompressible periodically distorted flow.

The mean potential flow around real airfoils is not uniform especially for lifting airfoils. Studying the interaction between a periodic two-dimensional gust with a lifting airfoil moving in incompressible flow, Goldstein & Atassi (1976) showed that the oncoming gust is distorted by the mean potential flow about the airfoil. This causes significant variation in both the amplitude and the phase of the unsteady velocity field associated with the gust. Their results, as well as the results obtained later by Atassi (1984) for cambered airfoils at non-zero angle of attack to the mean potential flow, show that the gust distortion has a significant effect on the airfoil response.

Another important example of streaming motions with small unsteady disturbances is provided by weakly turbulent flows around various bodies. It is again frequently possible to assume that the turbulent component of the flow is primarily distorted by changes in the mean flow, and to linearize the relevant motion with respect to a mean potential flow. This was first attempted by Prandtl (1933) and Taylor (1935). Later, Ribner & Tucker (1953) and Batchelor & Proudman (1954) studied the reduction in turbulence intensity due to a contraction in the stream. The latter introduced the term 'rapid distortion' to characterize this linear approach to the study of turbulent motions. Hunt (1973) generalized this theory to deal with incompressible flows around bluff bodies. He used the 'traditional splitting' of the velocity field into irrotational and solenoidal components and consequently was led to the mathematical problem of solving three Poisson's equations.

A unified approach to streaming motions with small unsteady disturbances imposed on the upstream flow was proposed by Goldstein (1978). He considered both vortical and entropic distortions of potential flows at arbitrary Mach number, and decomposed the unsteady velocity field into the sum of (i) a vortical disturbance $\boldsymbol{u}^{(V)}$ that is completely decoupled from the fluctuations in pressure or from any other thermodynamic property, and whose expression is a known function of the imposed upstream disturbance velocity, (ii) an entropy-dependent disturbance $\boldsymbol{u}^{(S)}$ whose expression is a known function of the imposed upstream entropy disturbance, and

(iii) an irrotational disturbance $\nabla\phi$ that produces no entropy fluctuations and is completely determined by a single inhomogeneous convective wave equation. The known expressions for $\boldsymbol{u}^{(V)}$ and $\boldsymbol{u}^{(S)}$ also involve the Lagrangian coordinates of the mean flow and their spatial gradients. The dipole source term in Goldstein's wave equation depends on the sum $\boldsymbol{u}^{(I)}$ of the first two disturbances and thus there is a partial coupling between the three components of the unsteady motion. Goldstein's decomposition of the velocity field reduces the mathematical problem for this kind of weakly distorted streaming motion to that of solving a single convected wave equation. This significantly simplifies the analytical treatments and the numerical procedures used to study such flows. For example, Goldstein's procedure leads to a single Poisson's equation for three-dimensional incompressible flows while Hunt's approach led to three such equations.

The present paper deals with this kind of weakly distorted streaming motion round bluff and streamlined bodies. The associated mean potential flow usually has a frontal stagnation point where the mean velocity vanishes and the Lagrangian coordinates and therefore $\boldsymbol{u}^{(I)}$ become singular and remain so along the entire body surface. Since the normal component of the total disturbance velocity must vanish at the body surface, the irrotational velocity $\nabla\phi$ must have a cancelling singular behaviour near the body surface. This makes it difficult to use Goldstein's approach directly to calculate this important class of flows numerically.

One purpose of the present paper is to modify Goldstein's partial splitting for flows with a leading stagnation point or line in a way that removes the singular and indeterminate character of the resulting boundary-value problem for such flows. Thus, it is shown that the disturbance velocity field can be split into (i) a part $\boldsymbol{u}^{(R)}$ whose expression is a known function of the upstream disturbances and whose normal and streamwise components vanish along the entire body surface and its wake, (ii) an entropy dependent disturbance whose expression is a known function of the imposed upstream entropy disturbances, and (iii) an irrotational part whose potential function satisfies Goldstein's wave equation with a modified source term.

For upstream periodic disturbances, explicit expressions for $\boldsymbol{u}^{(R)}$ are derived for a single obstacle in a three-dimensional flow and for a linear cascade of obstacles in a two-dimensional mean flow.

The present approach is also applied to the case of weakly sheared steady flows. After deriving the general expression for $\boldsymbol{u}^{(R)}$, it is shown that $\boldsymbol{u}^{(R)}$ is finite along the body surface when the mean flow has only a finite number of isolated stagnation points. On the other hand, when the mean flow has a stagnation line along the body surface such as in two-dimensional and certain axisymmetric flows, the component of $\boldsymbol{u}^{(R)}$ parallel to the stagnation line will have a logarithmic singularity along the entire body surface (Lighthill 1956). This singularity cannot be removed since the total perturbation velocity $\boldsymbol{u}$ itself has such a singularity.

Finally, in §4 the theory is applied to incompressible flows, in which case the governing equation reduces to a single Poisson's equation. Green's theorem is used to derive an integral equation for flows around bluff and streamlined bodies. The total unsteady velocity field is given in terms of quadratures for the case of a two-dimensional bluff body. Numerical calculations of the unsteady velocity and pressure fields are carried out for periodic disturbances around a typical bluff body particularly near the stagnation point.

2. Goldstein's splitting of the disturbance velocity

Consider a potential streaming motion around an obstacle and suppose that small vortical and entropic disturbances are imposed on the upstream flow. Let U_∞ be the constant upstream velocity and let $\boldsymbol{x} = \{x_1, x_2, x_3\}$ be Cartesian coordinates with the x_1-axis in the upstream mean velocity direction. Then far upstream ($x_1 \rightarrow -\infty$) the total velocity field $\boldsymbol{V}$ and the entropy S must be of the form

$$\boldsymbol{V} = \boldsymbol{i}U_\infty + \boldsymbol{u}_\infty(x_1 - U_\infty t, x_2, x_3), \tag{2.1}$$

$$S = s_\infty(x_1 - U_\infty t, x_2, x_3). \tag{2.2}$$

Here $\boldsymbol{i}$ is a unit vector in the x_1-direction and t is the time. $\boldsymbol{u}_\infty$ and $\boldsymbol{s}_\infty$ can be any functions of their arguments with the restriction that $\boldsymbol{u}_\infty$ be solenoidal,

$$\nabla \cdot \boldsymbol{u}_\infty = 0. \tag{2.3}$$

Goldstein (1978) showed that the resulting perturbation velocity $\boldsymbol{u} = \{u_1, u_2, u_3\}$ can be written at any point of the flow as

$$\boldsymbol{u} = \boldsymbol{u}^{(\mathrm{I})} + \nabla\phi, \tag{2.4}$$

where $\boldsymbol{u}^{(\mathrm{I})}$ is a rotational disturbance whose expression is a known function of the imposed upstream disturbances. $\nabla\phi$ is related to the perturbation pressure p' by

$$\frac{p'}{\rho_0} = -\frac{\mathrm{D}_0\phi}{\mathrm{D}t}, \tag{2.5}$$

where $\rho_0 = \rho_0(\boldsymbol{x})$ is the mean flow density and $\mathrm{D}_0/\mathrm{D}t$ is the convective derivative based on the mean flow velocity $\boldsymbol{U} = \{U_1, U_2, U_3\}$. $\boldsymbol{u}^{(\mathrm{I})} = \{u_1^{(\mathrm{I})}, u_2^{(\mathrm{I})}, u_3^{(\mathrm{I})}\}$ is given by

$$\boldsymbol{u}^{(\mathrm{I})} = \boldsymbol{u}^{(\mathrm{H})} + \frac{1}{2c_p} s_\infty(\boldsymbol{X} - \boldsymbol{i}U_\infty t)\,\nabla\Phi, \tag{2.6}$$

where Φ is the mean flow velocity potential, c_p is its specific heat at constant pressure (assumed to be constant) and $\boldsymbol{u}^{(\mathrm{H})} = \{u_1^{(\mathrm{H})}, u_2^{(\mathrm{H})}, u_3^{(\mathrm{H})}\}$ is given by

$$u_i^{(\mathrm{H})} = \boldsymbol{A}(\boldsymbol{X} - \boldsymbol{i}U_\infty t) \cdot \frac{\partial \boldsymbol{X}}{\partial x_i} \quad \text{for } i = 1, 2, 3, \tag{2.7}$$

where
$$\boldsymbol{A}(\boldsymbol{X} - \boldsymbol{i}U_\infty t) = \boldsymbol{u}_\infty(\boldsymbol{X} - \boldsymbol{i}U_\infty t) - \boldsymbol{i}(U_\infty/2c_p)\, s_\infty(\boldsymbol{X} - \boldsymbol{i}U_\infty t). \tag{2.8}$$

The components of the vector $(\boldsymbol{X} - \boldsymbol{i}U_\infty t)$ are essentially Lagrangian coordinates of the mean flow fluid particles. The components of $\boldsymbol{X} = \{X_1, X_2, X_3\}$ are defined as follows. $X_2(x_1, x_2, x_3)$ and $X_3(x_1, x_2, x_3)$ are independent integrals of the equations

$$\frac{\mathrm{d}x_1}{U_1} = \frac{\mathrm{d}x_2}{U_2} = \frac{\mathrm{d}x_3}{U_3} = \mathrm{d}t, \tag{2.9}$$

such that
$$X_2 \rightarrow x_2, \quad X_3 \rightarrow x_3 \quad \text{as } x_1 \rightarrow -\infty. \tag{2.10}$$

Thus the mean flow streamlines lie along the intersections of surfaces $X_2 = \text{constant}$ and $X_3 = \text{constant}$. The equations for the mean flow streamlines in terms of x_1 can be written as

$$x_2 = y_s(x_1, X_2, X_3), \quad x_3 = z_s(x_1, X_2, X_3). \tag{2.11}$$

X_1/U_∞ is the Lighthill (1956) 'drift' function

$$\Delta(x_1,x_2,x_3)=\frac{x_1}{U_\infty}+\int_{-\infty}^{x_1}\left[\frac{1}{U_1[x_1',y_s(x_1',X_2,X_3),z_s(x_1',X_2,X_3)]}-\frac{1}{U_\infty}\right]\mathrm{d}x_1'. \quad (2.12)$$

The change in Δ between any two points of a streamline is equal to the time it takes a mean flow fluid particle to traverse the distance between those two points.

Finally, the only unknown quantity is the perturbation potential ϕ which satisfies the linear inhomogeneous wave equation

$$\frac{\mathrm{D}_0}{\mathrm{D}t}\left(\frac{1}{c_0^2}\frac{\mathrm{D}_0\phi}{\mathrm{D}t}\right)-\frac{1}{\rho_0}\nabla\cdot(\rho_0\nabla\phi)=\frac{1}{\rho_0}\nabla\cdot\rho_0\boldsymbol{u}^{(\mathrm{I})}, \quad (2.13)$$

where $c_0=c_0(\boldsymbol{x})$ is the mean flow sound speed. For a rigid obstacle, the boundary condition at the obstacle surface Σ is

$$\boldsymbol{n}\cdot\nabla\phi=-\boldsymbol{n}\cdot\boldsymbol{u}^{(\mathrm{I})} \quad \text{for } \boldsymbol{x}\in\Sigma. \quad (2.14)$$

In addition

$$\phi(\boldsymbol{x},t)\rightarrow 0 \quad \text{as } x_1\rightarrow-\infty. \quad (2.15)$$

3. Modified splitting for flows round bodies

When the fluid motion is streaming around a body, there is usually a front stagnation point S on the body surface. In this case the 'drift' function Δ will develop a logarithmic singularity at S, that will remain along the entire body surface and its wake if any. Thus the argument $X_1-U_\infty t$ of the quantities introduced in (2.6)–(2.8) will be infinite. $\boldsymbol{u}_\infty(\boldsymbol{X}-\mathrm{i}\boldsymbol{u}_\infty t)$ and $s_\infty(\boldsymbol{X}-\boldsymbol{i}U_\infty t)$ will then be indeterminate for periodic disturbances. Moreover the rotational velocity $\boldsymbol{u}^{(\mathrm{I})}$ will have a reciprocal singularity along the entire body surface, since (2.7) shows that, in general $\boldsymbol{u}^{(\mathrm{H})}$ behaves as ∇X_1.

Turning to the boundary-value problem for the perturbation potential ϕ, we note that the right-hand side of (2.14) has, in general, a reciprocal singularity along the boundary. Then, recognizing that the total velocity $\boldsymbol{u}$ cannot have such a strong singularity on the body surface, we conclude that the dominant singular behaviour of $\boldsymbol{u}^{(\mathrm{I})}$ must be cancelled by $\nabla\phi$. These features make it difficult to use Goldstein's approach when numerically calculating the unsteady distorted flows over bodies with a stagnation point S.

In what follows we shall show that it is possible to find a potential velocity field that produces no pressure and that cancels the singular behaviour of the normal component of $\boldsymbol{u}^{(\mathrm{I})}$ along the body surface and its wake. The unsteady disturbance velocity can therefore be expressed as the sum of (i) a part that is a known function of the upstream disturbances and whose normal and streamwise components vanish along the entire body surface and its wake, (ii) an entropy dependent disturbance whose expression is a known function of the imposed upstream entropy disturbance, and (iii) an irrotational part whose potential function satisfies Goldstein's wave equation with a modified source term.

We start by analysing the behaviour of Δ near a stagnation point.

3.1. *The 'drift' function near a stagnation point*

Consider the mean potential flow around a surface Σ and let $S\in\Sigma$ be a stagnation point near which Σ is assumed to be smooth (figure 1). Let $(\boldsymbol{\tau},\boldsymbol{n},\boldsymbol{b})$ be the unit vectors of an orthogonal curvilinear coordinate system such that $\boldsymbol{\tau}$ is in the direction of the mean flow velocity, $\boldsymbol{n}$ in the direction of the outer normal to Σ and $\boldsymbol{b}=\boldsymbol{\tau}\times\boldsymbol{n}$. At S,

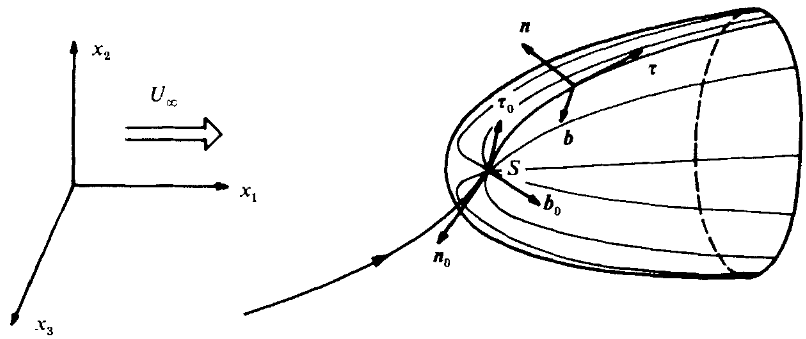

FIGURE 1. Coordinate system at the stagnation point.

this system is denoted $(\boldsymbol{\tau}_0, \boldsymbol{n}_0, \boldsymbol{b}_0)$. Let (s, n, b) be the coordinates of a point M near Σ with respect to $(\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b})$ and let (s_0, n_0, b_0) and (U_0, V_0, W_0) be, respectively, the coordinates of M and the components of the mean velocity $\boldsymbol{U}$ at M with respect to $(\boldsymbol{\tau}_0, \boldsymbol{n}_0, \boldsymbol{b}_0)$. Since at the stagnation point the two boundaries of the streamline cut at right angles, $\boldsymbol{n}_0$ is in the direction opposite to that of the mean flow velocity just upstream of S. As a result, the partial derivatives of V_0 with respect to s_0 and b_0 vanish at S and the leading term of the expansion of V_0 near S is given by

$$V_0 = \frac{n_0}{a_0} + \ldots, \tag{3.1}$$

where

$$a_0 = \left(\frac{\partial V_0}{\partial n_0}\right)_S^{-1} = -\left(\frac{\partial U}{\partial n_0}\right)_S^{-1}, \tag{3.2}$$

and $U = |\boldsymbol{U}|$. Substituting (3.1) into (2.12) to evaluate $\varDelta$ near S, we find

$$\varDelta = a_0 \ln n_0 + \tilde{\varDelta}(X_1, X_2, X_3), \tag{3.3}$$

where $\tilde{\varDelta}$ is a regular function of its arguments. Hence,

$$\frac{\partial \varDelta}{\partial n_0} = \frac{a_0}{n_0} + \ldots. \tag{3.4}$$

Thus, as a fluid particle near S moves close to Σ, the leading term in $\partial\varDelta/\partial n_0$ depends only on n_0. It follows immediately that as the fluid particles move close to Σ away from S, we can write,

$$\frac{\partial \varDelta}{\partial n} = \frac{a_0}{n} + \ldots. \tag{3.5}$$

For two-dimensional and axisymmetric flows, we can define a stream function $\Psi = \Psi(s, n)$ and rewrite (3.5) in terms of Ψ

$$\left(\frac{\partial \varDelta}{\partial \Psi}\right) = \frac{a_0}{\Psi - \Psi^{(0)}} + \ldots, \tag{3.6}$$

where $\Psi^{(0)}$ denotes the value of Ψ at Σ.

3.2. *A uniformly valid splitting of the velocity field*

We shall now show that it is possible to construct a potential velocity field that cancels the singular behaviour of the normal component of $\boldsymbol{u}^{(1)}$ along the body surface Σ and the wake $\mathscr{W}$. To this end, we first note that since $\nabla\Phi \cdot \boldsymbol{n} = 0$ on Σ and $\mathscr{W}$, (2.6) shows that the normal component of $\boldsymbol{u}^{(1)}$ is equal to that of the vortical velocity $\boldsymbol{u}^{(\mathrm{H})}$ which is a particular solution of the first-order homogeneous equation

$$\frac{\mathrm{D}_0}{\mathrm{D}t}\boldsymbol{u}^{(\mathrm{h})}+\boldsymbol{u}^{(\mathrm{h})}\cdot\boldsymbol{\nabla}U=0. \tag{3.7}$$

Equation (3.7) can be satisfied by a potential field $\boldsymbol{\nabla}\tilde{\phi}$ if

$$\frac{\mathrm{D}_0}{\mathrm{D}t}\tilde{\phi}=0, \tag{3.8}$$

which implies that $\tilde{\phi}=\tilde{\phi}(\boldsymbol{X}-\boldsymbol{i}U_\infty t)$. Note that the streamlines along the body surface are characteristics of (3.7) and (3.8). However, since both $\boldsymbol{u}^{(\mathrm{H})}$ and $\boldsymbol{\nabla}\tilde{\phi}$ are solutions of (3.7), a particular function $\tilde{\phi}$ can be constructed by imposing the additional condition that its normal gradient cancels that of $\boldsymbol{u}^{(\mathrm{H})}$ along Σ and $\mathscr{W}$, i.e.

$$(\boldsymbol{u}^{(\mathrm{H})}+\boldsymbol{\nabla}\tilde{\phi})\cdot\boldsymbol{n}=0 \quad \text{on } \Sigma+\mathscr{W}. \tag{3.9}$$

Equation (3.9) should be understood as the limit as we move close to Σ and $\mathscr{W}$. Note that since a streamline along Σ is a characteristic of (3.7), the boundary-value problem for $\tilde{\phi}$ is, in general, understated. Using (2.7), we write (3.9) as

$$\sum_{i=1}^{3}\left\{\left(A_i+\frac{\partial\tilde{\phi}}{\partial X_i}\right)\frac{\partial X_i}{\partial n}\right\}=0 \quad \text{on } \Sigma+\mathscr{W}, \tag{3.10}$$

where $\boldsymbol{A}=\{A_1,A_2,A_3\}$. Then since $\partial X_1/\partial n$ is singular along Σ and $\mathscr{W}$, (3.10) implies that

$$A_1+\frac{\partial\tilde{\phi}}{\partial X_1}=0 \quad \text{on } \Sigma+\mathscr{W}. \tag{3.11}$$

Since X_2 and X_3 are independent of s and $\partial X_1/\partial s=1/U$ (which is only singular at S), the streamwise component of $\boldsymbol{u}^{(\mathrm{H})}+\boldsymbol{\nabla}\tilde{\phi}$,

$$(\boldsymbol{u}^{(\mathrm{H})}+\boldsymbol{\nabla}\tilde{\phi})\cdot\boldsymbol{\tau}=\sum_{i=1}^{3}\left(A_i+\frac{\partial\tilde{\phi}}{\partial X_i}\right)\frac{\partial X_i}{\partial s}, \tag{3.12}$$

must vanish by virtue of (3.11), i.e.

$$(\boldsymbol{u}^{(\mathrm{H})}+\boldsymbol{\nabla}\tilde{\phi})\cdot\boldsymbol{\tau}=0 \quad \text{on } \Sigma+\mathscr{W}. \tag{3.13}$$

Thus the streamwise and normal components of the velocity field

$$\boldsymbol{u}^{(\mathrm{R})}=\boldsymbol{u}^{(\mathrm{H})}+\boldsymbol{\nabla}\tilde{\phi} \tag{3.14}$$

are not only non-singular but actually vanish at Σ and $\mathscr{W}$. Note that the $\boldsymbol{b}$-component of $\boldsymbol{u}^{(\mathrm{H})}$ may still have a singular or indeterminate behaviour depending on the functions A_1, A_2, A_3.

We therefore propose the following splitting of the disturbance velocity field

$$\boldsymbol{u}=\frac{1}{2c_p}s_\infty(\boldsymbol{X}-\boldsymbol{i}U_\infty t)\,\boldsymbol{\nabla}\Phi+\boldsymbol{u}^{(\mathrm{R})}+\boldsymbol{\nabla}\phi^*, \tag{3.15}$$

where $\boldsymbol{u}^{(\mathrm{R})}$ satisfies the equation

$$\frac{\mathrm{D}_0}{\mathrm{D}t}\boldsymbol{u}^{(\mathrm{R})}+\boldsymbol{u}^{(\mathrm{R})}\cdot\boldsymbol{\nabla}U=0, \tag{3.16}$$

and the boundary conditions on Σ and $\mathscr{W}$

$$\boldsymbol{u}^{(\mathrm{R})} \cdot \boldsymbol{\tau} = 0, \tag{3.17}$$

$$\boldsymbol{u}^{(\mathrm{R})} \cdot \boldsymbol{n} = 0. \tag{3.18}$$

The potential function ϕ^* satisfies the equation

$$\frac{\mathrm{D}_0}{\mathrm{D}t}\left(\frac{1}{c_0^2}\frac{\mathrm{D}_0}{\mathrm{D}t}\phi^*\right) - \frac{1}{\rho_0}\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\nabla}\phi^*) = \frac{1}{\rho_0}\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{u}^{(\mathrm{R})}) - \frac{1}{2c_p}\frac{\partial s_\infty}{\partial t}, \tag{3.19}$$

and the boundary conditions

$$\boldsymbol{\nabla}\phi^* \cdot \boldsymbol{n} = 0 \quad \text{on } \Sigma, \tag{3.20}$$

$$\Delta[\boldsymbol{\nabla}\phi^* \cdot \boldsymbol{n}] = 0 \quad \text{on } \mathscr{W}, \tag{3.21}$$

and

$$\boldsymbol{\nabla}\phi^* \to -\boldsymbol{\nabla}\tilde{\phi} \quad \text{as } x_1 \to -\infty, \tag{3.22}$$

where Δ denotes the jump across the wake.

It is important to note that for the determination of ϕ^*, (3.20) and (3.21) remove the major difficulty associated with the singular behaviour of X_1 along $\Sigma + \mathscr{W}$. Only the source term in (3.19) may be singular along $\Sigma + \mathscr{W}$. Moreover, the order of the singularity is lower than that of the source term in (2.13). These features make the present splitting particularly suitable for numerical calculations of subsonic and transonic (with weak shocks) flows. In this case, the mean potential flow is given by numerical codes and an accurate resolution of the flow quantities near the body surface is usually difficult to achieve. A numerical procedure based on the present method will not be very sensitive to the detail of the mean flow near the body surface.

We now turn to the determination of $\tilde{\phi}$. First, we note that the boundary conditions (3.17) and (3.18) are not equivalent to each other. It is easily seen that (3.18) always implies (3.17). The reciprocal is not true. We can satisfy (3.17) by taking $\tilde{\phi} = \tilde{\phi}_1$, where

$$\tilde{\phi}_1 = -\int^{X_1 - U_\infty t} A_1(x', X_2, X_3)\,\mathrm{d}x'. \tag{3.23}$$

The rotational velocity in the direction normal to Σ is

$$c_2\frac{\partial X_2}{\partial n} + c_3\frac{\partial X_3}{\partial n}, \tag{3.24}$$

where we have set

$$c_i = A_i + \frac{\partial\tilde{\phi}_1}{\partial X_i} \quad (i = 2, 3). \tag{3.25}$$

This removes the dominant singularity of $\boldsymbol{u}^{(\mathrm{H})}$ which is of the order of that of $\boldsymbol{\nabla}X_1$, but depending on the upstream conditions, c_i may still have a logarithmic singularity for sheared flows or be indeterminate for periodic disturbances. The singular behaviour of the component normal to Σ of the rotational velocity $\boldsymbol{u}^{(\mathrm{I})}$ is only completely eliminated when $\tilde{\phi}$ satisfies (3.18). We may write

$$\tilde{\phi} = \tilde{\phi}_1 + \tilde{\phi}_2, \tag{3.26}$$

where $\tilde{\phi}_2(\boldsymbol{X} - \boldsymbol{i}U_\infty t)$ is a function which, in view of (3.11) and (3.23), has a constant value along $\Sigma + \mathscr{W}$. Substituting (3.26) into (3.10), we arrive at

$$\sum_{i=1}^{3}\frac{\partial\tilde{\phi}_2}{\partial X_i}\frac{\partial X_i}{\partial n} = -\sum_{i=2}^{3} c_i\frac{\partial X_i}{\partial n}. \tag{3.27}$$

Noting from (3.11) and (3.23) that near Σ

$$\frac{\partial\tilde{\phi}_2}{\partial X_1} = \frac{\partial^2\tilde{\phi}_2}{\partial X_1 \partial X_2}\Delta X_2 + \frac{\partial^2\tilde{\phi}_2}{\partial X_1 \partial X_3}\Delta X_3 + \dots \tag{3.28}$$

and using (3.5) we arrive at

$$\frac{\partial\tilde{\phi}_2}{\partial X_1}\frac{\partial X_1}{\partial n} = a_0 U_\infty \sum_{i=2}^{3} \frac{\partial^2\tilde{\phi}_2}{\partial X_i \partial X_1}\frac{\partial X_i}{\partial n}. \tag{3.29}$$

Substituting (3.29) and (3.25) into (3.27) and using (3.23), we obtain after rearrangement

$$\sum_{i=2}^{3}\left\{\frac{\partial}{\partial X_i}\left[\left(\tilde{\phi} + a_0 U_\infty \frac{\partial\tilde{\phi}}{\partial X_1}\right) + a_0 U_\infty A_1\right] + A_i\right\}\frac{\partial X_i}{\partial n} = 0. \tag{3.30}$$

Equation (3.30) provides a boundary condition for $\tilde{\phi}$ on Σ. However, since this condition is given along a characteristic of (3.8), the boundary-value problem for $\tilde{\phi}$ is understated and the expression for $\tilde{\phi}$ cannot be entirely determined from (3.30). Nevertheless, we note that the quantity inside the brackets is a total differential, and since A_2 and A_3 are evaluated along Σ, we can therefore satisfy (3.30) near $X_2 = X_2^{(0)}$ and $X_3 = X_3^{(0)}$ by

$$\Delta\tilde{\phi} + a_0 U_\infty \frac{\partial \Delta\tilde{\phi}}{\partial X_1} = -a_0 U_\infty \Delta A_1 - A_2 \Delta X_2 - A_3 \Delta X_3, \tag{3.31}$$

where we have set

$$\Delta F \equiv F(X_1 - U_\infty t, X_2, X_3) - F(X_1 - U_\infty t, X_2^{(0)}, X_3^{(0)}). \tag{3.32}$$

$\Delta\tilde{\phi}$ may be analytically continued upstream of the body where the mean flow conditions are uniform. Thus, we obtain, after noting that $\tilde{\phi}_2$ may be taken to be zero along Σ,

$$\tilde{\phi} = -\int^{X_1 - U_\infty t} A_1(x', X_2^{(0)}, X_3^{(0)})\,\mathrm{d}x' - \exp\left(-\frac{X_1 - U_\infty t}{a_0 U_\infty}\right)\int^{X_1 - U_\infty t} \exp\left(\frac{x'}{a_0 U_\infty}\right) B(x', X_2, X_3)\,\mathrm{d}x', \tag{3.33}$$

where
$$B(X_1 - U_\infty t, X_2, X_3) = \Delta A_1 + \frac{1}{a_0 U_\infty}(A_2 \Delta X_2 + A_3 \Delta X_3). \tag{3.34}$$

Equation (3.33) is valid for any three-dimensional potential mean flow and any vortical and entropic upstream disturbances. The extension of the definition of $\tilde{\phi}$ to any X_2 and X_3 can be made to accommodate certain conditions for specific problems. For a mean flow past a cylinder with generator in the X_3-direction, we can take $X_2 = \Psi$, $X_3 = x_3$. Then $X_3^{(0)} = x_3$ and the last term in (3.34) completely vanishes.

3.3. *Periodic disturbances*

When the upstream disturbances are periodic, we can, without loss of generality, consider incident harmonic disturbances of the form

$$\boldsymbol{u}_\infty = \boldsymbol{a}\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{i}U_\infty t)] \quad \text{as } x_1 \to -\infty, \tag{3.35}$$

$$s_\infty = b\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{i}U_\infty t)] \quad \text{as } x_1 \to -\infty, \tag{3.36}$$

where $\boldsymbol{a} = \{a_1, a_2, a_3\}$, $\boldsymbol{k} = \{k_1, k_2, k_3\}$ and b are constants satisfying the continuity condition

$$\boldsymbol{a} \cdot \boldsymbol{k} = 0. \tag{3.37}$$

Substituting (3.35) and (3.36) into (2.8), we obtain

$$A(\boldsymbol{X} - \boldsymbol{i}U_\infty t) = \boldsymbol{a}^* \exp[\mathrm{i}\boldsymbol{k} \cdot (\boldsymbol{X} - \boldsymbol{i}U_\infty t)], \tag{3.38}$$

where we have set $\quad \boldsymbol{a}^* = \{a_1^*, a_2^*, a_3^*\} = \boldsymbol{a} - \boldsymbol{i}(bU_\infty/2c_p). \quad (3.39)$

Equation (3.33) can then be readily integrated and yields the following local expansion for $\tilde{\phi}$ near a streamline $X_2 = X_2^{(0)}$ and $X_3 = X_3^{(0)}$

$$\tilde{\phi} = \frac{\mathrm{i}}{k_1}\left\{A_1 - \frac{\Delta A_1 - \mathrm{i}k_1(A_2 \Delta X_2 + A_3 \Delta X_3)}{1 + \mathrm{i}a_0 U_\infty k_1}\right\} + \dots. \tag{3.40}$$

For a single obstacle we can extend $\tilde{\phi}$ for any X_2 and X_3 by taking

$$\tilde{\phi} = \frac{\mathrm{i}}{k_1}\left\{a_1^* + \frac{1}{1 + \mathrm{i}a_0 U_\infty k_1}\left[\frac{c_2^*}{k_2}(1 - \exp(-\mathrm{i}k_2(X_2 - X_2^{(0)}))) + \frac{c_3^*}{k_3}(1 - \exp(-\mathrm{i}k_3(X_3 - X_3^{(0)})))\right]\right\}\exp\{\mathrm{i}[\boldsymbol{k} \cdot (\boldsymbol{X} - \boldsymbol{i}U_\infty t)]\}, \tag{3.41}$$

where we have set $\quad c_i^* = (a_i^* k_1 - a_1^* k_i) \quad (i = 2, 3). \quad (3.42)$

For a mean flow past a cylinder with generator in the X_3-direction, $X_3 = X_3^{(0)} = x_3$. Thus the coefficient of c_3^* vanishes.

For a linear cascade with a spacing s^* perpendicular to the upstream mean flow, ϕ must vanish at all the cascade elements. This periodicity condition is satisfied by

$$\tilde{\phi} = \frac{\mathrm{i}}{k_1}\left\{a_1^* + \mathrm{i}\frac{s^* c_2^*}{2\pi(1 + \mathrm{i}a_0 U_\infty k_1)}\sin\left[\frac{2\pi}{s^* U_\infty}(\Psi - \Psi^{(0)})\right]\right\}\exp[\mathrm{i}\boldsymbol{k} \cdot (\boldsymbol{X} - \boldsymbol{i}U_\infty t)]. \tag{3.43}$$

Note that for periodic disturbances the component of $\boldsymbol{u}^{(\mathrm{R})}$ in the $\boldsymbol{b}$-direction will not be singular but will have an indeterminate phase.

3.4. *Weakly sheared flows*

The present theory can also be readily applied to steady potential flows with imposed upstream vortical and entropic disturbances. In this case, the upstream conditions (2.1) and (2.2) do not depend on time and as a result (2.8) depends only on X_2 and X_3. As an example, we consider a weakly sheared flow past an obstacle (figure 2) in which the upstream velocity field is a parallel flow

$$\boldsymbol{V} = \boldsymbol{i}U_\infty + (e_2 x_2 + e_3 x_3)\,\boldsymbol{i}, \tag{3.44}$$

where e_2 and e_3 are small constants. Then substituting the second term of the right-hand side of (3.44) into (2.8) and evaluating (2.7) we obtain

$$\boldsymbol{u}^{(\mathrm{I})} = (e_2 X_2 + e_3 X_3)\,\boldsymbol{\nabla} X_1. \tag{3.45}$$

The expression for $\tilde{\phi}$ is also readily obtained

$$\tilde{\phi} = -(e_2 X_2^{(0)} + e_3 X_3^{(0)})(X_1 - U_\infty t) - a_0 U_\infty[e_2(X_2 - X_2^{(0)}) + e_3(X_3 - X_3^{(0)})]. \tag{3.46}$$

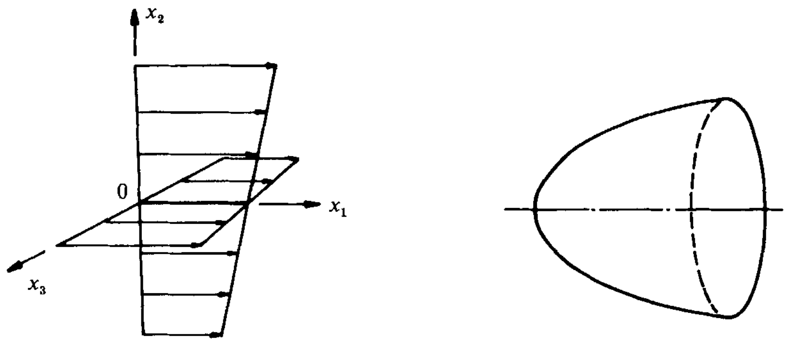

FIGURE 2. Body in a transversely sheared flow.

The expression for the velocity field $\boldsymbol{u}^{(\mathrm{R})}$ is obtained by adding (3.45) to the gradient of (3.46)

$$\boldsymbol{u}^{(\mathrm{R})} = -a_0 U_\infty[e_2 \nabla(X_2 - X_2^{(0)}) + e_3 \nabla(X_3 - X_3^{(0)})] - \left[e_2 \frac{\partial X_2^{(0)}}{\partial b} + e_3 \frac{\partial X_3^{(0)}}{\partial b}\right](X_1 - U_\infty t)\,\boldsymbol{b}$$
$$+ [e_2(X_2 - X_2^{(0)}) + e_3(X_3 - X_3^{(0)})]\nabla X_1, \quad (3.47)$$

$\boldsymbol{u}^{(\mathrm{R})}$ is the rotational velocity whose streamwise and normal components vanish at the surface of the obstacle and its wake.

We now examine the disturbance velocity component in the $\boldsymbol{b}$-direction. If the mean potential flow has only a finite number of isolated stagnation points, then $X_2^{(0)}$ and $X_3^{(0)}$ are constant and the rotational velocity (3.47) reduces to

$$\boldsymbol{u}^{(\mathrm{R})} = -a_0 U_\infty(e_2 \nabla X_2 + e_3 \nabla X_3) + [e_2(X_2 - X_2^{(0)}) + e_3(X_3 - X_3^{(0)})]\nabla X_1. \quad (3.48)$$

In general this velocity is *finite* and *not singular* in the $\boldsymbol{b}$-direction. On the other hand if the mean flow has a stagnation line along the surface such as in two-dimensional and certain axisymmetric flows, then $\partial X_2^{(0)}/\partial b$ and $\partial X_3^{(0)}/\partial b$ are, in general, not zero and as a result the disturbance velocity component in the $\boldsymbol{b}$-direction will be proportional to X_1 and thus will have a *logarithmic singularity* along the entire body surface.

For a two-dimensional mean flow about a cylinder with generator in the x_3-direction, we can take $X_2 = \Psi/U_\infty$, $X_2^{(0)} = \Psi^{(0)}/U_\infty$, $X_3 = X_3^{(0)} = x_3$. Thus (3.47) reduces to

$$\boldsymbol{u}^{(\mathrm{R})} = e_2\left[\frac{\Psi - \Psi^{(0)}}{U_\infty}\nabla X_1 - a_0 \nabla \Psi\right] - e_3(X_1 - U_\infty t)\,\boldsymbol{i}_3, \quad (3.49)$$

where $\boldsymbol{i}_3$ is a unit vector in the x_3-direction. The potential velocity field satisfies (3.19) and boundary conditions (3.20) to (3.22). It is often convenient to introduce a potential field that vanishes at infinity. This is done by writing

$$\phi^* = e_3(x_3/U_\infty)(\Phi - U_\infty^2 t) + e_2[\Psi^{(0)}(\Phi/U_\infty^2 - t - a_0) + a_0 \Psi] + \phi_1^*. \quad (3.50)$$

Then $\phi^* - \phi_1^* \to -\tilde{\phi}$ as $x_1 \to -\infty$. As a result, the boundary conditions for ϕ_1^* are

$$\frac{\partial \phi_1^*}{\partial n} = -e_2 a_0 \frac{\partial \Psi}{\partial n} \quad \text{for } \boldsymbol{x} \in \Sigma, \quad (3.51)$$

and $$\nabla \phi_1^* \to 0 \quad \text{as } x_1 \to -\infty. \quad (3.52)$$

Note that on Σ, $\partial \Psi/\partial n = \pm U$.

For an incompressible flow sheared only in the x_3-direction ($e_2 = 0$) it is easy to show that $\nabla \cdot \boldsymbol{u}^{(R)} = 0$, and that ϕ_1^* satisfies the Laplace's equation with homogeneous boundary conditions. Hence $\phi_1^* \equiv 0$. In this case the total disturbance velocity is given by

$$u_1 = e_3\left(\frac{x_3}{U_\infty}\right)\frac{\partial \Phi}{\partial x_1}, \tag{3.53}$$

$$u_2 = e_3\left(\frac{x_3}{U_\infty}\right)\frac{\partial \Phi}{\partial x_2}, \tag{3.54}$$

$$u_3 = e_3\,(\Phi/U_\infty - X_1), \tag{3.55}$$

which reproduces the result of Lighthill (1956).

4. Incompressible flows

When the mean flow Mach number is small and the frequency associated with the unsteady flow is not too large, the fluid motion is well approximated by that of an incompressible fluid. In this case $c_0 = \infty$ and $\rho_0 =$ constant. Equation (3.19) is thus reduced to a Poisson equation

$$\nabla^2\phi^* = -\nabla \cdot \left(\boldsymbol{u}^{(R)} + \frac{1}{2c_p} s_\infty\, \boldsymbol{U}\right), \tag{4.1}$$

with boundary conditions (3.20) to (3.22). It is however more convenient to introduce the function ϕ' whose gradient vanishes far upstream

$$\phi' = \phi^* + \tilde{\phi}_\infty, \tag{4.2}$$

where $\tilde{\phi}_\infty$ is a regular function such that

$$\nabla(\tilde{\phi} - \tilde{\phi}_\infty) \to 0 \quad \text{as } x_1 \to -\infty. \tag{4.3}$$

The unknown function ϕ' satisfies the equation

$$\nabla^2\phi' = -\nabla \cdot \boldsymbol{u}_0 \tag{4.4}$$

where we have set

$$\boldsymbol{u}_0 = \boldsymbol{u}^{(R)} + \frac{1}{2c_p} s_\infty\, \boldsymbol{U} - \nabla\tilde{\phi}_\infty. \tag{4.5}$$

Using (3.20) and (3.21), the boundary conditions for ϕ' are

$$\frac{\partial \phi'}{\partial n} = \frac{\partial \tilde{\phi}_\infty}{\partial n} \quad \text{on } \Sigma, \tag{4.6}$$

and

$$\frac{\partial \Delta\phi'}{\partial n} = \frac{\partial \Delta\tilde{\phi}_\infty}{\partial n} \quad \text{on } \mathscr{W}, \tag{4.7}$$

where $\Delta\phi'$ and $\Delta\tilde{\phi}_\infty$ stand for the jump of ϕ' and $\tilde{\phi}_\infty$ across the wake, and since far upstream $\boldsymbol{u}_0 \to \boldsymbol{u}_\infty$ and $\nabla \cdot \boldsymbol{u}_0 \to 0$, we can impose the condition

$$\phi' \to 0 \quad \text{as } x_1 \to -\infty. \tag{4.8}$$

4.1. *Integral formulation*

In many applications, it is more convenient to solve an integral equation numerically than a partial differential equation. To construct the integral equation for ϕ' with conditions (4.6)–(4.8), we first introduce the Green function $G(\boldsymbol{x}, \boldsymbol{x}_0)$ solution of the equation

$$\nabla_0^2 G = -m\pi\delta(\boldsymbol{x}_0 - \boldsymbol{x}), \tag{4.9}$$

where δ is Dirac's function, $m = 2$ for two-dimensional flows and $m = 4$ for three-dimensional flows, $\boldsymbol{x}_0$ is the observation point and $\boldsymbol{x}$ is the source point. It is shown in Appendix A that if the upstream disturbance $\boldsymbol{u}_\infty$ is such that the volume integral

$$\int_{\mathscr{V}} \boldsymbol{u}_\infty \cdot \boldsymbol{\nabla}_0 G \, \mathrm{d}v \tag{4.10}$$

exists, then ϕ' satisfies the following integral equation

$$\phi'(\boldsymbol{x}, t) = -\frac{1}{m\pi}\int_{\mathscr{V}} \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}_0 G \, \mathrm{d}v + \frac{1}{m\pi}\int_{\Sigma+\mathscr{W}} \phi' \frac{\partial G}{\partial n_0}\mathrm{d}\sigma - \frac{1}{m\pi}\int_{\mathscr{W}} G\left[\frac{\partial \phi'}{\partial n_0} - \frac{\partial \tilde{\phi}_\infty}{\partial n_0}\right]\mathrm{d}\sigma. \tag{4.11}$$

$\mathscr{V}$ is the volume outside Σ and $\mathscr{W}$. $\boldsymbol{\nabla}_0$ and $\partial/\partial n_0$ denote, respectively, the gradient and normal derivative with respect to the variable $\boldsymbol{x}_0$.

If the body extends to infinity downstream and there is no wake, the wake integrals in (4.11) vanish. In this case it may be advantageous to determine the Green function such that

$$\frac{\partial G}{\partial n_0} = 0 \quad \text{for } \boldsymbol{x}_0 \in \Sigma. \tag{4.12}$$

Then

$$\phi'(\boldsymbol{x}, t) = -\frac{1}{m\pi}\int_{\mathscr{V}} \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}_0 G \, \mathrm{d}v \tag{4.13}$$

and the problem can be solved by quadratures. In general, however, it is preferable to consider the free-space Green function since in this case

$$\Delta G = 0 \quad \text{for } \boldsymbol{x}_0 \in \mathscr{W}. \tag{4.14}$$

The last integral of (4.11) vanishes because of (4.7) and we have the following integral equation for ϕ',

$$\phi'(\boldsymbol{x}, t) = -\frac{1}{m\pi}\int_{\mathscr{V}} \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}_0 G \, \mathrm{d}v + \frac{1}{m\pi}\int_{\Sigma+\mathscr{W}} \phi' \frac{\partial G}{\partial n_0}\mathrm{d}\sigma. \tag{4.15}$$

Note that in spite of the singularity of G as $\boldsymbol{x}_0 \rightarrow \boldsymbol{x}$, in general, the integrals in (4.15) exist and therefore (4.15) is an integral equation of the Fredholm type. The jump of ϕ' across the wake can be obtained by using the condition that the pressure is continuous. For periodic disturbances of the form (3.35), we obtain

$$\Delta\phi' = \Delta\tilde{\phi}_\infty + C_0 \exp\left\{\mathrm{i}k_1 U_\infty\left(\int^s \frac{\mathrm{d}s}{U} - t\right)\right\}, \tag{4.16}$$

where C_0 is a constant to be determined by Kelvin's theorem.

4.2. *Flows past a bluff body*

As an illustration of the general theory we study the flow past a bluff body in the (x, y)-plane with emphasis on the flow behaviour near the stagnation point. We consider a relatively simple two-dimensional bluff-body shape extending to infinity with finite width. For such a body the steady mean flow is well approximated by a potential flow. For simplicity, we consider only two-dimensional disturbances. In this case, the expression for the Green function G satisfying (4.12) can be derived analytically and the total unsteady velocity field can be given in terms of quadratures of known functions. Let

$$W(z) = \Phi + \mathrm{i}\Psi \tag{4.17}$$

be the complex potential of the mean flow and $z = x + \mathrm{i}y$. We note that the transformation

$$T = W^{\frac{1}{2}} \tag{4.18}$$

maps the flow domain into the upper half of the T-plane. The body contour is mapped into the real axis. If we introduce the function

$$F(T_0, T) = -\ln\{(T_0 - T)(T_0 - \bar{T})\}, \tag{4.19}$$

where the overbar denotes the complex conjugate, then

$$G(T_0, T) = \mathrm{Re}\{F\} \tag{4.20}$$

is the Green function satisfying (4.12). The subscript 0 denotes the observation point. The expression for $\nabla_0 G$ which appears in (4.13) is then readily calculated in complex form,

$$\overline{\nabla_0 G} = -\left(\frac{\mathrm{d}T_0}{\mathrm{d}z_0}\right)\left[\frac{1}{T_0 - T} + \frac{1}{T_0 - \bar{T}}\right], \tag{4.21}$$

and the unknown function ϕ' is then given by

$$\phi' = -\frac{1}{2\pi}\iint_{\mathscr{S}} (\boldsymbol{u}_0 \cdot \nabla_0 G)\, \mathrm{d}x_0\, \mathrm{d}y_0, \tag{4.22}$$

where $\mathscr{S}$ is the area of the plane outside the body. The velocity $\nabla\phi'$ cannot, however, be calculated directly because any derivative of $\nabla_0 G$ with respect to x, y will lead to a non-integrable quantity. But $\nabla\phi'$ can be calculated following a method outlined by Tikhonov & Samerskii (1963, p. 376). Thus we obtain

$$\nabla\phi' = -\frac{1}{2\pi}\lim_{\epsilon\to 0}\iint_{\mathscr{S}-\mathscr{S}_\epsilon} \nabla(\boldsymbol{u}_0 \cdot \nabla_0 G)\, \mathrm{d}x_0\, \mathrm{d}y_0 - \frac{1}{2}\boldsymbol{u}_0. \tag{4.23}$$

$\mathscr{S}_\epsilon$ is a circle of radius ϵ centred at the point (x, y). When (x, y) belongs to the body surface the last term of (4.23) should be replaced by $-\frac{1}{4}\boldsymbol{u}_0$. Note that the numerical evaluation of the double integral in (4.23) must be carried out carefully as the small circle $\mathscr{S}_\epsilon$ is approached.

Finally, recall that the first term in (4.5) of $\boldsymbol{u}_0$ is the velocity $\boldsymbol{u}^{(\mathrm{R})}$ which has zero streamwise and normal components along the body surface but whose spanwise component may be singular or indeterminate. However, for a two-dimensional problem $\nabla_0 G$ has no component in the span direction. Therefore, the integrand in (4.23) is regular and well defined in $\mathscr{S} - \mathscr{S}_\epsilon$.

Periodic disturbances

For periodic disturbances of the form (3.35) and (3.36), we have the following expressions for $\boldsymbol{u}^{(\mathrm{R})}$ and $\tilde{\phi}_\infty$,

$$\boldsymbol{u}^{(\mathrm{R})} = \left\{\nabla\left[\left(\boldsymbol{a}^* - \frac{a_1^*}{k_1}\boldsymbol{k}\right)\cdot \boldsymbol{X}\right]E(\boldsymbol{X}) \right.$$
$$\left. + \frac{\mathrm{i}}{k_1}\frac{c_2^*}{k_2(1 + \mathrm{i}a_0 U_\infty k_1)}\nabla[E(\boldsymbol{X}) - E(\boldsymbol{X}^{(0)})]\right\}\exp(-\mathrm{i}k_1 U_\infty t), \tag{4.24}$$

$$\tilde{\phi}_\infty = \frac{\mathrm{i}}{k_1}\left\{a_1^* E(\boldsymbol{X}_\infty) + \frac{c_2^*}{k_2(1+\mathrm{i}a_0 U_\infty k_1)}[E(\boldsymbol{X}_\infty) - E(\boldsymbol{X}_\infty^{(0)})]\right\}\exp(-\mathrm{i}k_1 U_\infty t), \quad (4.25)$$

where we have put
$$\boldsymbol{X}_\infty = \frac{1}{U_\infty}(\Phi \boldsymbol{i} + \Psi \boldsymbol{j}), \quad (4.26)$$

$$X_2 = \Psi / U_\infty, \quad (4.27)$$

and
$$E(\boldsymbol{x}) = \exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x})\}. \quad (4.28)$$

The total unsteady velocity field,
$$\boldsymbol{u}_0 + \nabla\phi', \quad (4.29)$$

is obtained by substituting (4.24) and (4.25) into (4.5) and then evaluating the integral (4.23). Note that in the evaluation of (4.24), the accuracy is improved significantly by grouping the terms which cancel each other at the surface and thus avoid the inaccuracy resulting from the large phase variation.

We have not yet specified the shape of the body which of course enters the solution implicitly through G, $\boldsymbol{X}$ and $\boldsymbol{X}_\infty$. Expression (4.29) is therefore the most general solution for two-dimensional periodic disturbances convected by a potential flow without wake. Such potential flows represent a good approximation of real flows near the front part of a bluff body where the influence of the wake may be neglected (Hunt 1973). In the case of an airfoil with a sharp trailing edge, the aerodynamic forces are strongly influenced by the wake effect and will be studied in a forthcoming paper.

As an example we consider the flow generated by the superposition of a uniform flow U_∞ and a source at the origin of strength m. The complex potential for such a flow is
$$W = \Phi + \mathrm{i}\Psi = U_\infty z + m\ln z + C, \quad (4.30)$$

where C is an arbitrary constant. The equation for the body surface (figure 3) is given by
$$y = \frac{m}{U_\infty}\left(\pi - \tan^{-1}\frac{y}{x}\right). \quad (4.31)$$

It is convenient to choose the constant C in (4.30) such that W vanishes at the stagnation point. This leads to
$$C = m\left(1 - \ln\frac{m}{U_\infty}\right) - \mathrm{i}\pi m. \quad (4.32)$$

The steady complex velocity is given by
$$U_1 - \mathrm{i}U_2 = U_\infty + m/z. \quad (4.33)$$

As in many two-dimensional approximations, the potential flow velocity tends slowly to its upstream uniform value. To calculate the drift function Δ, one may not substitute U_1 from (4.33) into (2.12) because the integral in (2.12) does not converge. This difficulty can be removed by assuming that at a large distance from the stagnation point we have a finite body and that the velocity tends to its upstream uniform value as $1/|z|^2$ (Atassi 1984). However, it is simpler to modify the definition of Δ, to
$$\Delta = \frac{x}{U_\infty} + \int_{x_0}^{x}\left[\frac{1}{U_1(x', y'(x', \Psi))} - \frac{1}{U_\infty}\right]\mathrm{d}x', \quad (4.34)$$

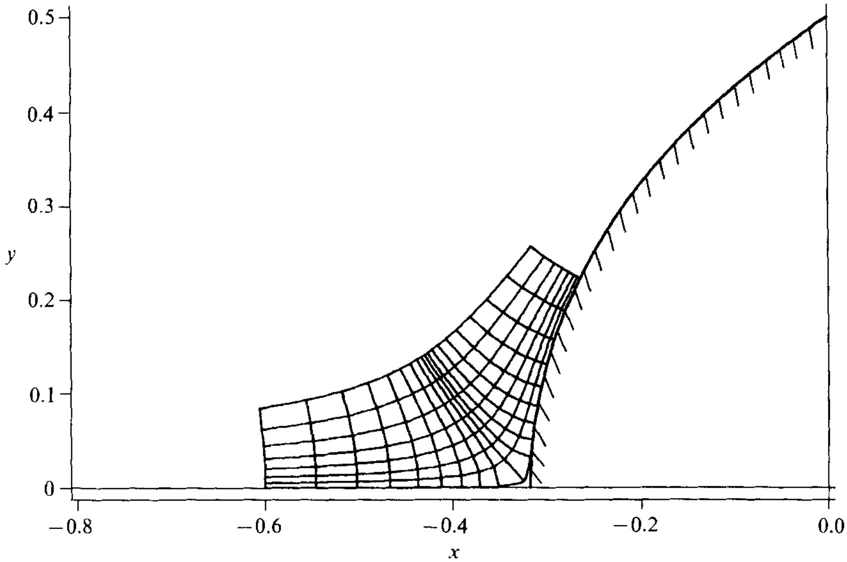

FIGURE 3. The steady flow around a typical bluff body near the stagnation point.

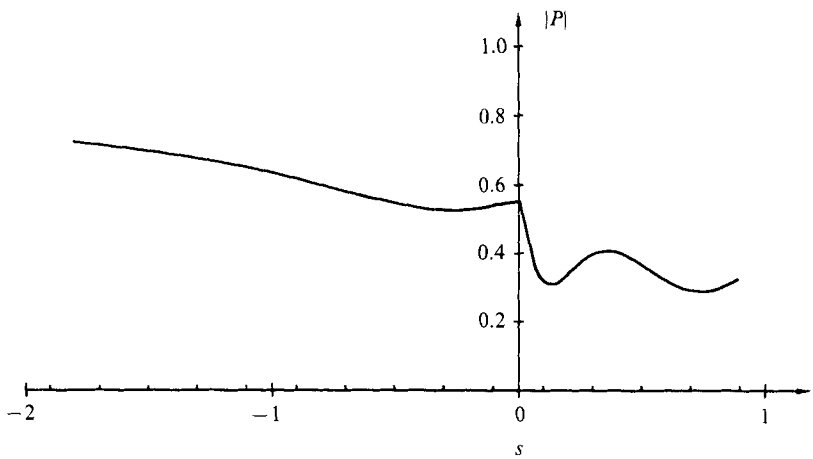

FIGURE 4. Variation of the magnitude of the normalized unsteady pressure along the stagnation streamline for a typical bluff body subject to an upstream harmonic disturbance with $k_1 = k_2 = 1$.

where x_0 is the abscissa of a point $M_0(x_0, y_0)$ upstream and such that $|x_0| \gg m/U_\infty$. Equation (4.34) can be readily evaluated noting that $\mathrm{d}x/U_1 = |\mathrm{d}z/(U_1 - \mathrm{i}U_2)|$, and we get

$$\Delta = \frac{x}{U_\infty} + \frac{m}{U_\infty^2} \ln \left| \frac{z_0 + m/U_\infty}{z + m/U_\infty} \right|. \tag{4.35}$$

Note that by fixing x_0, we introduce a constant phase difference in the gust which, for $|x_0| \gg m/U_\infty$, will not influence the basic solution (Goldstein 1978).

Figure 3 shows the geometry of the body. As the body extends downstream, its half-thickness is $h = \pi m/U_\infty$. The upstream harmonic disturbance is characterized by the two wave numbers k_1 and k_2. Both are non-dimensionalized with respect to

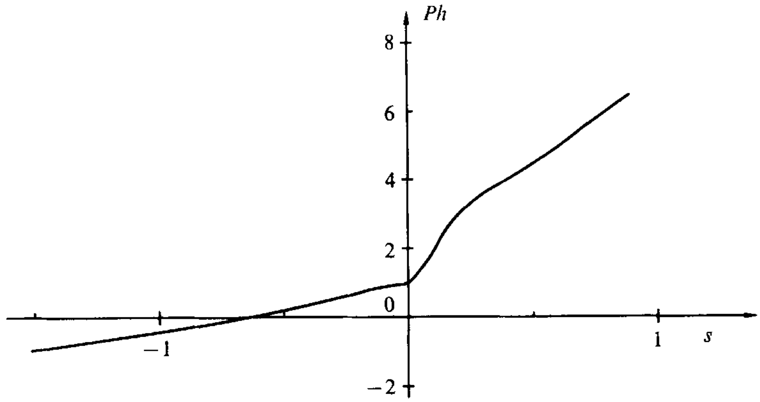

FIGURE 5. Variation of the phase of the normalized unsteady pressure along the stagnation streamline for a typical bluff body subject to an upstream harmonic disturbance with $k_1 = k_2 = 1$.

h. Detailed calculations were carried out for various k_1 and k_2. The unsteady velocity and pressure fields were calculated at the intersection of the lines Φ = constant and Ψ = constant. The area of integration for (4.23) is a square in the (Φ, Ψ)-plane, whose side is equal to about $40h$. As expected from the theory, the calculated streamwise and normal components of the vortical velocity $\boldsymbol{u}^{(\mathrm{R})}$ vanish at the body surface. The total unsteady velocity is finite at the stagnation point. Figures 4 and 5 show the magnitude and the phase of the normalized unsteady pressure $P = p'/(\rho_0 U_\infty a)$ versus the distance s along the stagnation streamline for $h = 1$, $k_1 = 1$ and $k_2 = 1$. The pressure is, of course, continuous but the pressure gradient has a discontinuity at the stagnation point.

Appendix A

Consider a surface Σ_∞ surrounding the body Σ and intersecting its wake $\mathscr{W}$ as shown in figure 6. Let $\boldsymbol{n}$ be the outward unit normal to Σ, $\mathscr{W}$ and Σ_∞, and let $\mathscr{R}$ be the region enclosed by Σ, $\mathscr{W}$ and Σ_∞. We denote by $\boldsymbol{n}'$ the unit normal on Σ, $\mathscr{W}$ and Σ_∞, directed outward of the region $\mathscr{R}$. Thus $\boldsymbol{n}' = \boldsymbol{n}$ on Σ_∞ and $\boldsymbol{n}' = -\boldsymbol{n}$ on $\Sigma + \mathscr{W}$. The subscript 0 will denote these vectors in the $\boldsymbol{x}_0$-space. Applying Green's theorem to the functions ϕ' and G and using (3.18), (4.4)–(4.7) and the divergence theorem, we obtain after rearrangement

$$\phi'(\boldsymbol{x}, t) = -\frac{1}{m\pi}\int_{\mathscr{R}} \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}_0 G \,\mathrm{d}v - \frac{1}{m\pi}\int_{\Sigma+\mathscr{W}+\Sigma_\infty} \phi' \frac{\partial G}{\partial n_0'}\,\mathrm{d}\sigma + \frac{1}{m\pi}\int_{\mathscr{W}+\Sigma_\infty} G[(\boldsymbol{u}_0 + \boldsymbol{\nabla}_0\phi')\cdot\boldsymbol{n}_0']\,\mathrm{d}\sigma. \quad (\mathrm{A}\ 1)$$

Let ρ_{m} be the minimum distance of $\boldsymbol{x}$ to Σ_∞. Then noting that as $\rho_{\mathrm{m}} \to \infty$, $|\partial G/\partial n|\,\mathrm{d}\sigma$ is of the order of the elementary solid angle under which $\mathrm{d}\sigma \in \Sigma_\infty$ is seen from $\boldsymbol{x}$, and since $\phi' \to 0$ as $\rho_{\mathrm{m}} \to \infty$ except maybe near the wake, we conclude that

$$\int_{\Sigma_\infty} \phi' \frac{\partial G}{\partial n'}\,\mathrm{d}\sigma \to 0 \quad \text{as}\ \rho_{\mathrm{m}} \to \infty, \quad (\mathrm{A}\ 2)$$

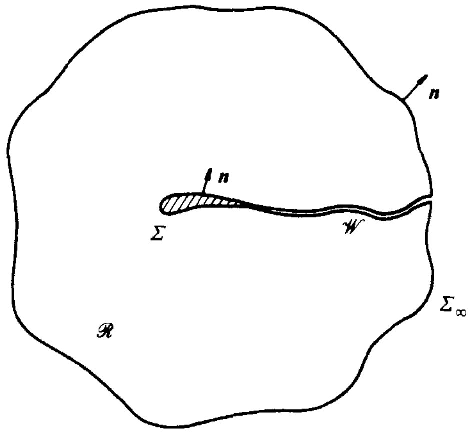

FIGURE 6. Schematic of the region $\mathscr{R}$ used in the derivation of the integral equation (A 8).

We now assume that the upstream disturbance $\boldsymbol{u}_\infty$ is such that

$$\int_{\mathscr{V}} (\boldsymbol{u}_\infty \cdot \boldsymbol{\nabla}_0 G)\,\mathrm{d}v \tag{A 3}$$

exists. Since $\boldsymbol{\nabla} \cdot \boldsymbol{u}_\infty = 0$, (A 3) implies that as $\rho_{\mathrm{m}} \rightarrow \infty$

$$\int_{\Sigma_\infty} G(\boldsymbol{u}_\infty \cdot \boldsymbol{n}_0)\,\mathrm{d}\sigma \tag{A 4}$$

is finite and its value independent of Σ_∞. Since Σ_∞ is an arbitrary surface, we can take Σ_∞ to be a sphere centred on $\boldsymbol{x}$. In this case, G is constant and can be factored out of the integral in (A 4) which then vanishes because of the divergence theorem. Thus,

$$\int_{\Sigma_\infty} G(\boldsymbol{u}_\infty \cdot \boldsymbol{n}_0)\,\mathrm{d}\sigma \rightarrow 0 \quad \text{as } \rho_{\mathrm{m}} \rightarrow \infty. \tag{A 5}$$

Since $\boldsymbol{u}_0 + \boldsymbol{\nabla}_0 \phi'$ is also solenoidal and tends to $\boldsymbol{u}_\infty$ at large distance except maybe near the wake, (A 3) also implies that

$$\int_{\mathscr{V}} [(\boldsymbol{u}_0 + \boldsymbol{\nabla}_0 \phi') \cdot \boldsymbol{\nabla}_0 G]\,\mathrm{d}v \tag{A 6}$$

exists. Following the same proof as for (A 5), we get

$$\int_{\Sigma_\infty} G[(\boldsymbol{u}_0 + \boldsymbol{\nabla}_0 \phi') \cdot \boldsymbol{n}_0]\,\mathrm{d}\sigma \rightarrow 0 \quad \text{as } \rho_{\mathrm{m}} \rightarrow \infty. \tag{A 7}$$

Therefore as $\rho_{\mathrm{m}} \rightarrow \infty$, and using (3.18) and (4.5), (A 1) reduces to:

$$\phi'(\boldsymbol{x}, t) = -\frac{1}{m\pi}\int_{\mathscr{V}} \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}_0 G\,\mathrm{d}v + \frac{1}{m\pi}\int_{\Sigma+\mathscr{W}}^{\phi'} \frac{\partial G}{\partial n_0}\,\mathrm{d}\sigma - \frac{1}{m\pi}\int_{\mathscr{W}} G\left[\frac{\partial \phi'}{\partial n_0} - \frac{\partial \tilde{\phi}_\infty}{\partial n_0}\right]\mathrm{d}\sigma. \tag{A 8}$$

It is convenient for evaluating the volume integral in (A 8) to add (A 3) and to use the divergence theorem to obtain the following expression for the integral equation for ϕ'

$$\phi'(\boldsymbol{x},t) = -\frac{1}{m\pi}\int_{\mathscr{V}}(\boldsymbol{u}_0-\boldsymbol{u}_\infty)\cdot\boldsymbol{\nabla}_0 G\,\mathrm{d}v+\frac{1}{m\pi}\int_{\Sigma+\mathscr{W}}\phi'\frac{\partial G}{\partial n_0}\mathrm{d}\sigma$$
$$+\frac{1}{m\pi}\int_{\Sigma+\mathscr{W}}G(\boldsymbol{u}_\infty\cdot\boldsymbol{n}_0)\,\mathrm{d}\sigma-\frac{1}{m\pi}\int_{\mathscr{W}}G\left[\frac{\partial\phi'}{\partial n_0}-\frac{\partial\tilde{\phi}_\infty}{\partial n_0}\right]\mathrm{d}\sigma. \quad (\mathrm{A}\ 9)$$

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